Structured Stability Robustness Improvement by Eigenspace Techniques: A Hybrid Methodology

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A new hybrid design technique for improving stability robustness with respect to block-structured perturbations in multivariable linear feedback systems is presented. This new methodology is hybrid in that it uses both constrained optimization techniques for robustness and eigenstructure assignment for performance. This approach allows the designer to treat simultaneous perturbations occurring at different locations in the feedback system without having to compromise between robustness and performance. Performance requirements expressed in terms of closed-loop eigenvalues and right-eigenvectors are directly satisfied by the procedure. The realistic example of the SA365N DOLPHIN helicopter is presented to highlight the merits and some of the concerns in using this methodology.

Nomenclature

I	= identity matrix
j	$= \sqrt{-1}$
P_i	= weighting matrix associated to V_i
s	= Laplace variable
$V_i = V(\lambda_i)$	= ith closed-loop eigenvector
V_i^d	= desired eigenvectors
X_δ, X_∞	= sets of uncertainty
$\Delta\lambda_i$	= small variation on ith eigenvalue
$\Delta(s)$	= perturbations on the nominal system
δ	= uncertainty bound
λ_i	= ith closed-loop eigenvalue
μ	= robustness measure
ρ	= spectral radius
$\underline{\rho}$	= smallest modulus of eigenvalues
σ_i	= ith singular value
$\bar{\sigma}, \underline{\sigma}$	= maximum and minimum singular value
ω	= frequency, rad/s
det( )	= determinant of ( )
Im(.)	= range of the map (.)
Re(.)	= real part of (.)
$\mathbb{R}_+$	= set of positive real numbers
$\mathbb{C}^{k \times k}$	= set of $k \times k$ complex matrices
$(\ )^+$	= complex conjugate transpose of ( )
$(\ )^*$	= complex conjugate of ( )

Introduction

In recent years there has been a growing interest in the synthesis of robust controllers. Because of the considerable design experience with classical optimal linear-quadratic (LQ) and linear-quadratic Gaussian (LQG) design, there was strong interest in extending robustness results to this class of problems, and a multivariable robust design philosophy emerged. It was identified as the linear-quadratic Gaussian/loop transfer recovery (LQG/LTR) approach (see Doyle[1] and Stein[2]). More recently, new design methods employing constrained optimization techniques to search for fixed-structure controller design parameters that improve stability robustness measures expressed in terms of singular values, were developed (see Mukhopadhyay and Newson[3,4]). These methodologies have to face two types of difficulties. On the one hand, stability and performance measures based on singular values are generally too conservative for structured perturbations of the nominal system. On the other hand, the choices of design parameters such as weighting and noise matrices, and the a priori structure of compensators are not straightforward, especially if our goal is to adequately solve the essential tradeoffs of feedback design between robustness and performance.

The purpose of this paper is to introduce a new design method in order to cope with these difficulties. Therefore, a design synthesis is developed to handle block-structured perturbations at different locations in the system. Stability robustness in regard to block-structured perturbations is considered via the structured singular value derived by Doyle.[5,6] The technique is elaborated on the basis of the eigenstructure assignment procedure, which has proven to be well-suited for obtaining specified input/output-response properties,[17] although this ability is not followed by any significant robustness guarantees comparable to LQ guarantees.

Structured and Unstructured Analysis of Stability Robustness

Any linear interconnection of inputs, outputs, commands, perturbations, and a controller can be rearranged in the form of Fig. 1. Nominal model $P(s)$, controller $K(s)$, and perturbations $\Delta(s)$ are assumed throughout to be linear time invariant; $\boldsymbol{v}$ and $\boldsymbol{y}$ are input and output external signal vectors. Because we are concerned with stability robustness analysis, the diagram of Fig. 1 reduces to that of Fig. 2. The analysis model isolates the nominal part $M(s)$ of the system from the perturbations $\Delta(s)$. The transfer-function matrix $M(s)$ includes compensation $K(s)$ and so is assumed stable.

In this form, stability analysis essentially boils down to ensuring that $I + M(s)\Delta(s)$ remains nonsingular at all frequencies and for all perturbations $\Delta(s)$ under consideration. There is a great disparity of robustness measures in the literature. See, for example, Barrett,[7] Kantor,[8] and Lehtomaki.[9] In this paper we have chosen the μ analysis tool developed by Doyle[6] because it is well adapted to our problem, particularly in the presence of structured uncertainties, as we shall see in the sequel. We begin by defining the general framework of robustness analysis.

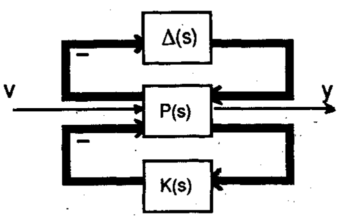

Fig. 1 General model.

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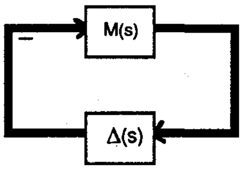

Fig. 2 Stability robustness analysis model.

According to Doyle,[6] structured sets of perturbations are considered in the form

$$X_\delta = [\text{block-diag}(\Delta_1, \Delta_2, \dots \Delta_n) \mid \Delta_i \in \mathbb{C}^{ki \times ki}, \quad \bar{\sigma}(\Delta_i) \le \delta, i = 1, \dots n] \quad (1)$$

These sets are called structured because they require information about their internal block-diagonal structure, whereas the elementary blocks Δ_i are assumed unstructured because we have information on their norm only. We denote by X_∞ all such Δ matrices with no restriction on the norm. Then

$$X_\infty = \bigcup_{\delta \in \mathbb{R}_+} X_\delta \quad (2)$$

Let

$$k = \sum_{i=1,n} k_i \quad (3)$$

where n is the number of the square blocks Δ_i of dimension k_i.

With these notations the μ measure can be defined as a map from $\mathbb{C}^{k \times k}$ to $\mathbb{R}_+$

$$\mu: \mathbb{C}^{k \times k} \longrightarrow (0, +\infty)$$

such that

$$\det(I + M\Delta) \neq 0$$

for all $\Delta \in X_\delta$, if and only if

$$\delta \mu(M) < 1 \quad (4)$$

Or alternatively, μ could be defined as

$$\mu^{-1}(M) = \begin{cases} \infty \text{ if no } \Delta \in X_\infty \text{ solves } \det(I + M\Delta) = 0 \\ \min\limits_{\Delta \in X_\infty} [\bar{\sigma}(\Delta) / \det(I + M\Delta) = 0] \end{cases} \quad (5)$$

Then we can state the following theorem in a slightly different manner than presented by Doyle.[6]

μ-theorem The analysis model Fig. 2 will remain stable in the presence of block-structured perturbations $\Delta(s)$ such that $\Delta(s)$ belongs to X_δ for all $s = j\omega$, and $\Delta(s)$ stable if and only if

$$\sup_\omega \mu[M(j\omega)] < \frac{1}{\delta} \quad (6)$$

(See Doyle[5,6] for more details and demonstration.)

Therefore, this theorem provides necessary and sufficient stability conditions for structured uncertainties. It is important to note that all simultaneous uncertainties can always be put into block-diagonal form by constructing the corresponding transfer-function matrix $M(s)$. The structured singular value μ enables designers to treat and analyze uncertainties occurring simultaneously at several different locations in the feedback control system; μ is nonconservative in the sense that it takes into account the internal structure of $\Delta(s)$. Unfortunately, the optimization problem described in Eq. (5) is generally nonconvex and so computation of μ is complicated in nature. This problem has been the center of a great amount of research recently and is still computationally complicated (see Safonov[11–13] for completeness).

However, μ has several useful and interesting properties from a practical point of view. Some of these are interpreted below without proof.[5]

Fundamental properties:

For perturbations Δ structured by X_∞, we have

$$\bar{\sigma}(M) \ge \mu(M) \quad (7)$$

for all M in $\mathbb{C}^{k \times k}$. If $n = 1$, Δ is an unstructured $k \times k$ block; then

$$\mu(M) = \bar{\sigma}(M) \quad (8)$$

for all M in $\mathbb{C}^{k \times k}$.

Interpreted in robustness terms, the inequality (7) highlights the conservativeness of robustness tests based on singular values when perturbations $\Delta(s)$ are block-structured. On the other hand, equality (8) means that in the absence of structured information, singular value and structured singular value join together.

Scalar-type perturbations:

If $X_\delta = [\lambda I / \lambda \in \mathbb{C}, \mid \lambda \mid \le \delta]$, then

$$\mu(M) = \rho(M) \quad (9)$$

for all M in $\mathbb{C}^{k \times k}$.

Therefore, from a robustness point of view, the spectral radius of the interconnection matrix $M(s)$ may be used when perturbations are scalar type, namely $\Delta(s) = \lambda(s) I$. This property justifies the use of characteristic gains defined by McFarlane[14] for evaluating stability margins of the feedback system. However, from Eq. (9) these margins are valid only for scalar-type perturbations and not for true multivariable perturbations.

Fundamental bounds on the structured singular value:

Let $\mathscr{D}$ be the set of block-structured diagonal matrices:

$$\mathscr{D} = [\text{diag}(d_1 I_{k1}, d_2 I_{k2}, \dots, d_n I_{kn}) / d_i \in \mathbb{R}] \quad (10)$$

Let $\mathscr{U}$ be the set of block-structured unitary matrices:

$$\mathscr{U} = [\text{block-diag}(U_1, U_2, \dots, U_n) \mid U_i \in \mathbb{C}^{ki \times ki}, \quad U_i^+ \times U_i = I_{ki}] \quad (11)$$

Then we have the interesting property

$$\max_{U \in \mathscr{U}} \rho(UM) \le \mu(M) \le \min_{D \in \mathscr{D}} \bar{\sigma}(DMD^{-1}) \quad (12)$$

for all M in $\mathbb{C}^{k \times k}$.

It was shown that the left-hand side inequality is always an equality but the optimization problem involved has multiple local maxima that may be nonglobal. The right-hand side inequality is an equality only for three or fewer elementary blocks Δ_i, and the minimization problem involved is convex in D so that there exists no local minimum that is not global. Furthermore, computational experiment has shown that min $\bar{\sigma}(DMD^{-1})$ over $\mathscr{D}$ may be, in most cases, considered as a good approximation of μ for more than three blocks. So, we make this assumption in the remainder of this paper and choose to work with min $\bar{\sigma}(DMD^{-1})$ over $\mathscr{D}$, which is denoted $\bar{\mu}$ and called the pseudostructured singular value (*PSSV*). We shall show that our assumption is valid.

Computation of the PSSV

For robustness analysis, structured or unstructured, we need to compute $\bar{\mu}$ or $\bar{\sigma}$. The computation of $\bar{\sigma}$ is straightforward by available numerical software (Eispack or Linpack). For $\bar{\mu}$, we have to solve a convex nonlinear optimization problem. In order to implement a nonlinear programming technique, we have to determine the gradients of $\bar{\sigma}(DMD^{-1})$ with respect to D. It was shown in Ref. 15 that for a complex matrix $A(\alpha)$ analytic with respect to the real parameter α, distinct singular values of $A(\alpha)$ are Frechet-differentiable, and the Frechet derivative with respect to α is given by

$$\frac{\partial}{\partial \alpha}\sigma_1(\alpha) = \mathrm{Re}\left\{U_l + \frac{\partial}{\partial \alpha}[A(\alpha)]V_l\right\} \tag{13}$$

where V_l and U_l are, respectively, right and left normalized singular vectors of $A(\alpha)$ associated with $\sigma_l(\alpha)$. Using Eq. (13) we obtain

$$\frac{\partial \bar{\sigma}}{\partial d_i}(DMD^{-1}) = \mathrm{Re}(U^+\hat{I}_iMDV)/d_i^2 + \mathrm{Re}(U^+D^{-1}M\hat{I}_iV) \qquad i = 1,\ldots n \tag{14}$$

where

$$\hat{I}_i = \text{block-diag}\ (0\,0\,0\,0\ldots 0\ I_{ki}\ 0,\ldots 0) \tag{15}$$

V,U are singular vectors associated with $\bar{\sigma}(DMD^{-1})$.

Therefore, the PSSV $\bar{\mu}$ can be calculated by a nonlinear programming technique using analytical gradients. We tried two methods for solving μ: conjugate gradients and BFGS (Broyden, Fletcher, Goldfarb and Shanno).[24] The latter seems very well suited to our problem.

Eigenstructure Assignment and First Design Methodology

In the above section we have focused on stability analysis by recalling the fundamental structured singular value (SSV). We also have introduced a method with analytical gradients to compute an approximate value (PSSV). So we now have to design synthesis methods in order to improve this fundamental quantity.

Here, we choose to use the freedom inherent in eigenstructure assignment as a synthesis tool. We begin by discussing briefly the problem of simultaneous assignment of a set of closed-loop eigenvalues and eigenvectors using state feedback.

Consider the linear, time invariant system described by

$$\dot{x} = Ax + Bu \tag{16}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and rank$(B) = m$ and (A,B) is a controllable pair.

Given a set $\Lambda = \{\lambda_i\}_{i=1,n}$, self-conjugate, of distinct complex numbers containing no open-loop eigenvalues, then the following is true[16–19] (see Fig. 3):

Theorem 1. There exists a real $(m \times n)$ state feedback matrix K such that

$$(A + BK)V_i = \lambda_i V_i \qquad i = 1,2\ldots n \tag{17}$$

if and only if 1)

$$V = (V_i)_{i=1,n} \tag{18}$$

are linearly independent in $\mathbb{C}^n$, 2)

$$V_i = V_j^* \tag{19}$$

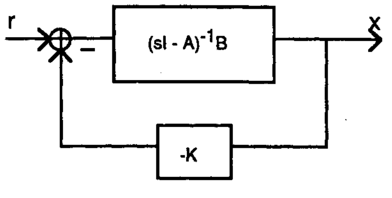

Fig. 3 State feedback.

when $\lambda_j^* = \lambda_i$, and 3)

$$V_i \in (\lambda_i I - A)^{-1}B \tag{20}$$

The importance of this theorem is that each right-eigenvector V_i must reside in subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$. This subspace is generically of dimension m, which is the number of independent control variables. The orientation of the subspace is governed by the open-loop parameters A, B and the desired closed-loop eigenvalues $\{\lambda_i\}_{i=1,n}$.

If we denote

$$V_i = (\lambda_i I\text{-}A)^{-1}\cdot BW_i, \qquad i = 1,\ldots n \tag{21}$$

where $W = (W_i)_{i=1,n}$ are the input vectors, then the state-feedback law is given by

$$K = W \cdot V^{-1} \tag{22}$$

The design parameters of the modal synthesis are then the input vectors W_i and the closed-loop desired eigenvalues $\{\lambda_i\}_{i=1,n}$.

Therefore, a first design methodology should merely solve

$$\text{minimize} \sup_{\omega} \bar{\mu}[M(j\omega)] \tag{23}$$

with respect to λ_i and W_i, or equivalently, if M is invertible,

$$\text{maximize} \inf_{\omega} \underline{\mu}[M^{-1}(j\omega)] \tag{24}$$

with respect to λ_i and W_i, where $\underline{\mu}$ denotes here

$$\max_{D\in\mathscr{D}} \bar{\sigma}(D^{-1}M^{-1}D)$$

$M(s)$ is the analysis matrix derived from Fig. 3, depending on the locations of the simultaneous perturbations $\Delta_i(s)$.

Note that when the perturbation is unstructured we can always derive analytical gradients of $\underline{\sigma}(M^{-1})$ or $\bar{\sigma}(M)$ with respect to λ_i and W_i.

When it is structured such quantities do not exist because $\bar{\mu}(M)$ or $\underline{\mu}(M^{-1})$ is not a function in a classical sense but rather a solution of an optimization problem. We have to use finite differences to evaluate corresponding gradients.

This first methodology has several shortcomings that we develop here briefly (see Refs. 20–22). Although this method allows the designer to directly control the system dynamics via the eigenvalues, some performance properties are nevertheless lost during the optimization procedure. Indeed, if the procedure is initialized by right-eigenstructure assignment, altering input vectors generally implies a very large change of corresponding eigenvectors followed by a change of the response shaping. Such a problem is amplified when the system requires a high degree of decoupling. Similar problems were raised by Newson and Mukhopadhyay.[3,4] An increase in robustness often is accompanied by degraded response and increased control activity. Thus, the designer has to compromise between robustness and performance.

Another difficulty may be the choice of eigenvalues and input vectors that have to be optimized first in order to preserve some aspects of performance like decoupling and modal contributions. Furthermore, this first methodology is based on the entire available freedom $(\lambda_i, W_i)_{i=1,n}$, so the designer has a great number of parameters to optimize and there may be a lot of trial and error before obtaining the desired feedback qualities. Therefore, this brings us to a second design methodology that avoids the above-mentioned difficulties.

Second Hybrid Robust Design Methodology

The purpose of this section is to construct a versatile methodology that allows the designer to control the two essential objectives of feedback, performance, and stability robust-

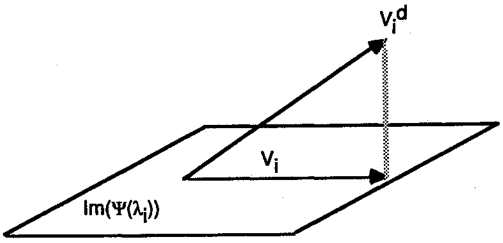

Fig. 4 Performance requirements: P_i-orthogonal projection on characteristic subspaces.

ness. The former is expressed in the time domain in terms of the eigenvalues and right-eigenvalues of the closed-loop system. The latter is expressed in the frequency domain in terms of the pseudostructured singular value calculated with the adequate analysis model.

Given the controllable system described in Eq. (16), we know that for a fixed self-autoconjugate set of complex numbers $\{\lambda_i\}_{i=1,n}$, corresponding right-eigenvectors must reside in the subspace spanned by the columns of $(\lambda_i I - A)^{-1} \cdot B$, and so they can be partially assigned with m entries in each. For time domain response shaping, input vectors w_i may be selected to minimize the weighted sum of the squares of the difference between the elements of the desired and attainable eigenvectors as given by the following performance index:

$$J_i = (V_i - V_i^d)^+ P_i(V_i - V_i^d), \qquad i = 1,2, \ldots n \qquad (25)$$

where V_i^d are the desired right-eigenvectors condensing the objectives like decoupling and modes repartition; P_i are positive definitive symmetric matrices whose elements can be chosen to weight the difference between certain elements of the desired and attainable eigenvectors more heavily than others.

Defining

$$\Psi(\lambda_i) = (\lambda_i I - A)^{-1} \cdot B \qquad (26)$$

right eigenvectors are constrained by

$$V_i = \Psi(\lambda_i) W_i \qquad (27)$$

Solutions to our minimization problems are immediately given by P_i-orthogonal projection on Im[$\Psi(\lambda_i)$], hence

$$W_i = [\Psi^+(\lambda_i) P_i \Psi(\lambda_i)]^{-1} \Psi^+(\lambda_i) P_i V_i^d, \qquad i = 1,2, \ldots n \qquad (28)$$

Geometric interpretation is depicted in Fig. 4. Note that we do not separate real problems from complex ones, as in the literature, because they can be solved in the same way. Therefore, for fixed spectra $\Lambda = \{\lambda_i\}_{i=1,n}$, performance requirements are simply obtained by Eq. (28).

We now have to take into account the robustness of the feedback control system, which is expressed by the appropriate analysis model $M(s)$, depending on the location of the perturbations. In order to do that we use the fact that, on the one hand, stability robustness is intimately related to the relative locations of modes in the left-half complex plane, whereas on the other hand, for practical synthesis the designer never needs to accurately place the closed-loop eigenvalues, but rather to constrain them into more or less extended areas. Consequently, the procedure consists in generating small variations $\Delta\lambda_i$ on closed-loop eigenvalues to improve stability robustness and then, for each infinitesimal variation of the spectra Λ, to make P_i-orthogonal projections according to Eq. (28). A geometric interpretation of one step of the procedure is depicted in Fig. 5. It is shown that $\Delta\lambda_i$ variations imply modifications on the orientation of corresponding characteristic subspaces Im[$\Psi(\lambda_i)$].

This second methodology is hybrid in the sense that it uses a nonlinear programming technique for stability robustness and point-wise well-known modal synthesis for performance.

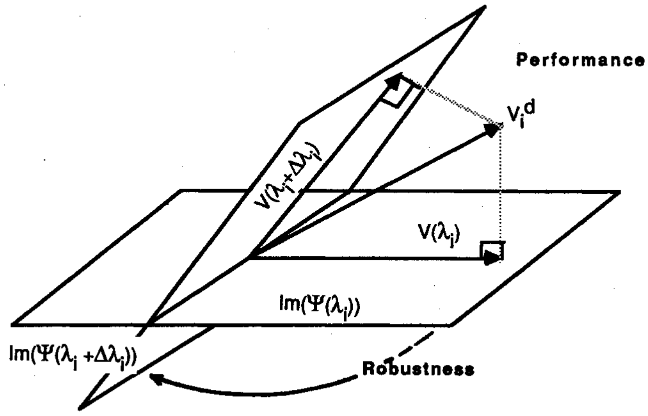

Fig. 5 Robustness and performance requirements.

Therefore, the method requires a reduced number of parameters, which here are the closed-loop desired eigenvalues. In addition, the designer does not have to trade off performance requirements against robustness because performance is automatically guaranteed by the procedure. However, the method has the same drawbacks of any optimization technique. It is not guaranteed to converge and the obtained solution is very dependent on initial design.

In the first step, the designer must analyze the stability problem in terms of perturbation locations, level of structure, frequency range of interest, etc. In the second step he must express the corresponding stability robustness analysis model (see Refs. 21 and 22 for details).

The third step is relative to performance objectives like decoupling, mode distribution, acceptable damping, and settling time. So, rectangular areas are delimited in the complex plane. The last step of the analysis part is the choice of frequency range and number of frequencies for optimization. They can be found, for instance, by simulation of an initial design. Then, the designer can use the hybrid robust methodology that requires the computation of $\bar{\mu}$ or $\bar{\sigma}$ for structured or unstructured uncertainties.

The next section will highlight the merits and some of the concerns in using the hybrid methodology by illustrating a realistic nonacademic example.

Design Example

In order to demonstrate the hybrid robust methodology outlined above, a numerical example will be presented. The intent of this example is to illustrate the efficiency of the methodology on a strong performance-demanding plant. So, we consider the SA365 Dolphin Helicopter (see Ref. 23) described by Eq. (16).

The flight condition corresponds to a longitudinal velocity of 75 km/h. The eighth-order plant state vector is defined as:

u = longitudinal velocity (m/s)
w = normal velocity (m/s)
q = pitch rate (deg/s)
θ = pitch angle (deg) (29)
v = lateral velocity (m/s)
p = roll rate (deg/s)
ϕ = bank angle (deg)
r = yaw rate (deg/s)

The system's inputs are:

δ_2 = longitudinal cyclic pitch control input (deg)
δ_0 = collective pitch control input (deg)
δ_1 = lateral cyclic pitch control input (deg) (30)
δ_r = tail rotor pitch control input (deg)

We need to augment the given plant with additional dynamics, for example, to achieve integral action in the system. So, the augmented state and corresponding augmented state equa-

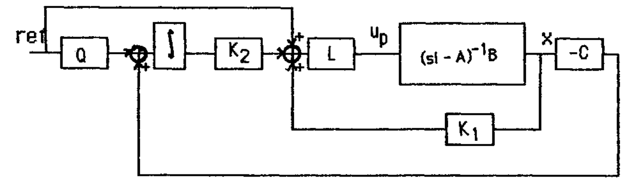

Fig. 6 Localization of uncertainties.

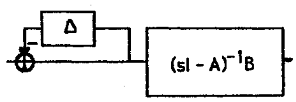

Fig. 7 Alternative representation of perturbation.

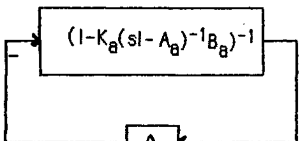

Fig. 8 Helicopter stability robustness analysis model.

tions are given by

$$\dot{x}_a = A_a x_a + B_a u \tag{31}$$

$$x_a = \begin{bmatrix} x \\ \int - u \, dt \\ \int - w \, dt \\ \int - v \, dt \\ \int - r \, dt \end{bmatrix} \tag{32}$$

with

$$A_a = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \qquad B_a = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{33}$$

and $-C$ associated with the chosen integrals.

Steps 1 and 2

Uncertainties are considered at the plant input as shown in Fig. 6. [$L(s)$ is the transfer matrix of uncertainties.] The state feedback control law with integral action is defined as

$$u = (K_1 K_2) x_a + \text{ref} \tag{34}$$

$$K_a = (K_1 K_2) \tag{35}$$

where K_1 and K_2 denote proportional and integral feedback gains, ref is the reference input signal, u_p is the perturbed plant input signal, and Q is a diagonal matrix for controlling the steady-state amplitudes of u, w, v, r.

The relation between perturbed and unperturbed plant inputs can be written as

$$u_p = Lu = (\Delta + I)^{-1} u \tag{36}$$

where

$$\Delta = L^{-1} - I \tag{37}$$

Then perturbations can be represented as shown in Fig. 7. Isolating the nominal part of the feedback control system from the perturbation Δ, we obtain the stability analysis model of Fig. 8.

In order to take into account parasitic effects of actuators and delays due to discretizing at the plant input, perturbations are considered in the form

$$L = \text{diag}(k_i e^{j\theta_i})_{i=1,\ldots,n} \tag{38}$$

Perturbations are then diagonally structured with simultaneous and independent gain and phase margins k_i and θ_i in every loop. The choice of a diagonal structure avoids considering physically nonexistent coupling problems at the plant input. From the μ-theorem section above [Eq. (6)], a nonconservative stability robustness condition can be expressed as

$$\inf_{\omega} (\mu(M^{-1}(j\omega)) > \bar{\sigma}(\Delta) \tag{39}$$

with

$$M(s) = (I - K_a(sI - A_a)^{-1} B_a)^{-1} \tag{40}$$

where it can be shown that

$$\bar{\sigma}(\Delta) = \bar{\sigma}(L^{-1} - I) = \max_i \sqrt{(1 - 1/k_i)^2 + 2(1 - \cos\theta_i)/k_i} \tag{41}$$

The term μ is calculated by the well-suited *BFGS*'s method with analytical gradients from Eq. (14).

Denoting

$$\mu_{\min} = \inf_{\omega} (\mu(M^{-1}(j\omega))) \tag{42}$$

with nominal phases $\theta_i = 0$, then structured upward gain margins are at least as large as

$$\frac{1}{1 - \mu_{\min}} \tag{43}$$

Structured reduction gain margins are at least as small as

$$\frac{1}{1 + \mu_{\min}} \tag{44}$$

With nominal gains $k_i = 1$, then structured phase margins are given by

$$\pm 2\text{Arcsin}\left(\frac{\mu_{\min}}{2}\right) \tag{45}$$

This means that gains k_i can vary independently and simultaneously in the above intervals without destabilizing the system when phases are nominal, and vice versa for phases.

Table 1 Modes distribution and desired eigenvectors

Motion	Longitudinal		Lateral		Normal	Turn
Modes	$\lambda_1,\lambda_2 = \lambda_3^*$ Integral λ_4		$\lambda_5,\lambda_6 = \lambda_7^*$ Integral λ_8		λ_9 Integral λ_{10}	λ_{11} Integral λ_{12}
Desired eigenvectors	x	x	0	x	0	0
	0	x	0	x	x	0
	x	x	0	x	0	0
	x	x	0	x	0	0
	0	x	x	x	0	0
	0	x	x	x	0	x
	0	x	x	x	0	x
	0	x	0	x	0	x
	x	x	x	0	0	x
	x	0	x	0	x	x
	x	0	x	x	0	x
	x	0	x	0	0	x

Table 2 Lateral constraints: initial and final design

Motion	Longitudinal	Lateral	Normal	Turn
Lateral constraints	$-1.1 \le \lambda_1 \le -0.47$ $-0.98 \le \lambda_2^r \le -0.32$ $0.01 \le \lambda_2^i \le 0.62$ $-0.53 \le \lambda_4 \le -0.03$	$-1.41 \le \lambda_5 \le -0.54$ $-1.78 \le \lambda_6^r \le -0.7$ $0.1 \le \lambda_6^i \le 0.85$ $-0.61 \le \lambda_8 \le -0.06$	$-2.0 \le \lambda_9 \le 0.86$ $-1.0 \le \lambda_{10} \le -0.47$ —— ——	$-2.2 \le \lambda_{11} \le -1.37$ $-1.39 \le \lambda_{12} \le -0.76$ —— ——
Initial modes	-0.52 $-0.38 \pm j0.45$ -0.05	-0.61 $-0.69 \pm j0.82$ -0.08	-1.0 -0.5	-1.8 -0.9
Final modes	-0.71 $-0.51 \pm j0.27$ -0.36	-1.29 $-1.62 \pm j0.1$ -0.46	-1.84 -1.05	-1.51 -1.72

Steps 3 and 4

In the third and fourth step, the designer has to choose the desired eigenstructure and the weighting matrices to accomplish orthogonal projections on characteristic subspaces. For the helicopter problem the state vector is adequately partitioned as follows:

u,q,θ, and $\int -u\,dt$ are associated with longitudinal motion;
w and $\int -w\,dt$ are associated with normal motion;
v,p,ϕ, and $\int -v\,dt$ are associated with lateral motion;
r,p,ϕ, and $\int -r\,dt$ are associated with coordinated turn.

These partitioning and decoupling considerations lead to the desired eigenstructure given in Table 1 where the symbol x denotes unspecified components of the eigenvectors. In accordance with these, weighting matrices P_i are chosen with no weight on the unspecified components and a weight equal to 1 for prescribed ones. The desired closed-loop dynamics of the feedback control system are shown in Table 2. Closed-loop modes and corresponding lateral constraints are chosen according to desirable transient behavior and settle time.

Step 5 Synthesis

The hybrid robust methodology was applied to our problem. The design synthesis was initiated with eigenvalues presented in Table 2 and corresponding eigenvectors obtained by P_i-orthogonal projections. The four elementary motions of the SA365N DOLPHIN helicopter are decoupled, but the system has poor robustness properties in regard to diagonal structured perturbations ($\underline{\mu}_{\min} = 0.26$).

In Fig. 9 the three curves represent $\underline{\rho}[M^{-1}(j\omega)]$, $\underline{\mu}[M^{-1}(j\omega)]$, and $\underline{\sigma}[M^{-1}(j\omega)]$ for the initial design; $\underline{\rho}(M^{-1})$ is the adequate measure for scalar-type perturbations; $\underline{\mu}(M^{-1})$ is the pseudostructured singular value that is an approximation of the perfect measure $\mu^{-1}[M(j\omega)]$; $\underline{\sigma}[M^{-1}(j\omega)]$ corresponds to unstructured perturbations.

Hence, we have the following inequalities:

$$\underline{\sigma}[M^{-1}(j\omega)] \le \underline{\mu}[M^{-1}(j\omega)] \le \mu^{-1}[M(j\omega)] \le \underline{\rho}[M^{-1}(j\omega)] \tag{46}$$

for all ω.

Eqs. (43)–(45) yield the following initial margins:

1.35(2.6dB) upward gain margins,
0.793(−2dB) gain reduction margins,
± 15 deg phase margins.

The system has poor robustness near $\omega = 1.3$ rad/s, and so we decided to optimize

$$J = \min \underline{\mu}[M^{-1}(j\omega)] \tag{47}$$

and $\omega = 0.2, 1.3, 2.0$ rad/s. These frequencies have been chosen in order to control the critical low-frequency interval.

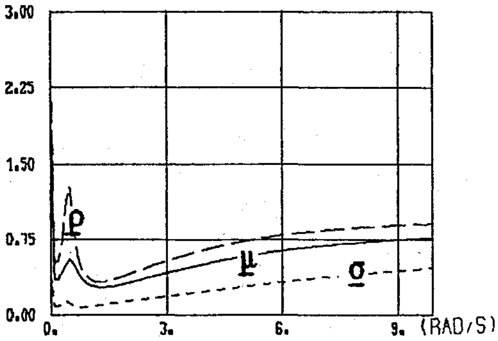

Fig. 9 Initial robustness measures.

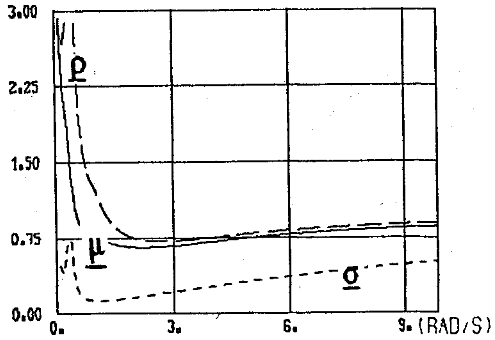

Fig. 10 Final robustness measures.

The optimization process converged at the fifth iteration and the three abovementioned robustness measures are shown in Fig. 10. The final multivariable margins are:

3.34 (10.48 dB) upward gain margins,
0.588 (−4.9 dB) gain reduction margins,
±41 deg phase margins.

Corresponding closed-loop eigenvalues are presented in Table 2. The hybrid method provides a significant increase in diagonal margins at the input of the plant. Note that initial and final design are characterized by very close infimum relative to $\underline{\sigma}(M^{-1})$ (Figs. 9 and 10). Hence, unstructured stability robustness has not been improved by the hybrid method. This result highlights the inherent conservatism of singular values when perturbations are structured. This brings us to ask if methods based only on singular values are well-suited when designers have some information on how the system is perturbed. Note also that the infimum of $\underline{\mu}(M^{-1})$ and $\underline{\rho}(M^{-1})$ with respect to frequency are not too far apart.

Hence, $\mu^{-1}(M)$, the ideal measure of robustness that is situated between them [Eq. (46)], is well approximated by $\underline{\mu}(M^{-1})$, and this validates our initial assumption in the above section.

Therefore, the final design is characterized by good robustness properties. Damping, settling time, and decoupling are nominally perfect because eigenvalues have been constrained and eigenvectors adequately chosen.

Conclusion

It has been shown that a hybrid technique combining eigenvalue/eigenvector assignment with optimization of a structured singular value measure can be used in the design of robust state feedback for flight control systems. This new approach optimizes robustness while preserving performance, and allows designers to meet settling time, damping, decoupling, and stability robustness requirements at the same time.

References

[1]Doyle, J. C. and Stein, G., "Multivariable Feedback Designs: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control*, Vol. 26, 1981, pp. 4–16.

[2]Stein, G. and Athans, M., "The LQG/LTR Procedure for Multivariable Feedback Control Design," *IEEE Transactions on Automatic Control*, Vol. 32, 1987, pp. 105–114.

[3]Newson, J. R. and Mukhopadhyay, V., "Design Study Using Singular Value Gradients," *AIAA Journal*, 1983, pp. 514–519.

[4]Newson, J. R. and Mukhopadhyay, V., "The Use of Singular Values Gradients and Optimization Techniques to Design Robust Controllers for Multiloop Systems," *AIAA Journal*, pp. 173–180.

[5]Doyle, J. C., "Analysis of Feedback Systems with Structured Uncertainties," *IEEE Proceedings Conference on Decision and Control*, Vol. 129, 1982, pp. 242–250.

[6]Doyle, J. C., "Performance and Robustness Analysis for Structured Uncertainty," *IEEE Proceedings Conference on Decision and Control*, Vol. 129, 1982, pp. 629–636.

[7]Barrett, M. F., "Conservatism with Robustness Test for Linear Feedback Control Systems," *IEEE Proceedings Conference on Decision and Control*, 1980, pp. 885–890.

[8]Kantor, J. C., "The Analysis of Robust Stability and Performance in Multivariable Feedback Systems Using M-Matrices; "*IEEE Proceedings Conference on Decision and Control*, Vol. 129, 1985, pp. 1309–1314.

[9]Lehtomaki, N. A., Castanon, D., Levy, B., Stein, G., Sandell, N. R., and Athans, M., "Robustness Tests Utilizing the Structure of Modelling Error," *IEEE Proceedings Conference on Decision and Control*, 1981, pp. 1173–1189.

[10]Lehtomaki, N. A., "Robustness Results in Linear Quadratic Gaussian Based Multivariable Control Designs," *IEEE Transactions on Automatic Control*, Vol. 26, 1983, pp. 75–92.

[11]Safonov, M. G. and Doyle, J. C., "Minimizing Conservativeness of Robustness Singular Values," *Multivariable Control*, edited by S. G. Tzafestas, Dordrecht, Holland, pp. 197–207.

[12]Safonov, M. G., "Stability Margins of Diagonally Perturbed Multivariable Feedback Systems," *IEEE Proceedings*, 1982, pp. 251–256.

[13]Safonov, M. G., "Exact Calculation of the Multivariable Structured Singular Value Stability Margin," *IEEE Proceedings Conference on Decision and Control*, 1984, pp. 1224–1225.

[14]MacFarlane, A. G. J. and Postlewaite, I., "The Generalized Nyquist Stability Criterion and Multivariable Root Loci," *International Journal of Control*, Vol. 25, 1977, pp. 81–127.

[15]Freudenberg, J. S., Looze, D. P., and Cruz, J. B., "Robustness Analysis Using Singular Value Sensitivities," *International Journal of Control*, Vol. 25, Jan. 1982, pp. 95–116.

[16]Moore, B. C., "On the Flexibility Offered by State Feedback in Multivariable Systems," *IEEE Transactions on Automatic Control*, Vol. 32, 1976, pp. 689–692.

[17]Shapiro, E. Y., Andry, A. N., and Chung, J. C., "Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Automatic Control*, Vol. 19, 1983, pp. 711–728.

[18]Magni, J. F., "Commande par Placement de Directions Propres," Contrat STPA N° 8495016, Automatic Division, Cert, France, March 1985.

[19]Bsalis, E., "Eigenstructure Assignment via Mode Participation Concept," internal report, Automatic Division, Cert, France, 1985.

[20]Apkarian, P., "Presentation de la Stabilite Externe dans le Cadre de l'Analyse en Robustesse en Stabilite des Systemes Multivariables," Contrat DRET No. 85/1469, Automatic Division, Cert, France, 1987, No. 1.

[21]Apkarian, P., "Analyse Structuree et non Structuree de Robustesse en Stabilite," Contrat DRET N°. 85/1469, Automatic Division, Cert, France, 1987, No. 2.

[22]Apkarian, P., "Synthese Hybride Robuste," Contrat DRET No. 85/1469, Automatic Division, Cert, France, 1987, No. 3.

[23]Gimonet, B., "Commande Selon un Modele de Reference: Application au Decouplage et a la Maniabilite d'un Helicoptere," *Commande Numerique des Systemes*, edited by Masson, Paris, 1986, Chap. 12, pp. 320–336.

[24]Shanno, D. F. and Phua, K. H., "Minimization of Unconstrained Multivariate Functions," *ACM Transactions on Mathematical Software*, Vol. 2, No. 1, March 1976, pp. 87–94.